JOURNAL OF COMBINATORIAL THEORY, Series B 23, 139–142 (1977)

# Hamiltonian Cycles in Regular Graphs of Moderate Degree

PAUL ERDÖS

*Hungarian Academy of Sciences, Budapest, Hungary*

AND

ARTHUR M. HOBBS

*Department of Mathematics, Texas A & M University, College Station, Texas 77843*

Received October 25, 1976

In this paper we prove that if $k$ is an integer no less than 3, and if $G$ is a two-connected graph with $2n - a$ vertices, $a \in \{0, 1\}$, which is regular of degree $n - k$, then $G$ is Hamiltonian if $a = 0$ and $n \geqslant k^2 + k + 1$ or if $a = 1$ and $n \geqslant 2k^2 - 3k + 3$.

We use the notation and terminology of [1]. Gordon [4] has proved that there are only a small number of exceptional graphs with $2n$ vertices which are not Hamiltonian when all vertices have degree $n - 1$ or more. The present authors proved [3] that if $G$ is a two-connected graph with $2n$ vertices which is regular of degree $n - 2$ and if $n \geqslant 6$, then $G$ is Hamiltonian. We now partially extend that result to regular graphs of degree $n - k$, $k \geqslant 3$.

Throughout this paper we suppose that $G$ is a graph with $2n - a$ vertices, with $a \in \{0, 1\}$, which is two-connected and regular of degree $n - k$, where $k$ is an integer no less than three. Let $P$ be a longest cycle in $G$, choose a direction around $P$, let $R = V(G) - V(P)$, and let $r = |R|$. For the lemmas, suppose $r \geqslant 1$. By a theorem of Dirac [2], $\ell(P) \geqslant 2n - 2k$. For $v \in R$, let $C_v$ be the set of vertices of $P$ adjacent to $v$, let $A_v$ be the set of vertices of $P$ immediately preceding elements of $C_v$ in the ordering of $P$, and let $B_v$ be the set of vertices of $P$ immediately following elements of $C_v$. The first lemma is trivial.

LEMMA 1. *Let $v$ and $w$ be in $R$. Then $v$ is not adjacent to any vertex in $A_v \cup B_v$, $A_v$ and $B_v$ are independent sets of vertices, and $w$ is joined to at most one vertex of $A_v$ and to at most one vertex in $B_v$.*

LEMMA 2. *If $n \geqslant 3k + 2 - a$, then $R$ is independent.*

*Proof.* Let $Q$ be a longest path in a component of $R$ and suppose $\ell(Q) \geqslant 1$. Let $v$ and $w$ be the ends of $Q$ and let $d = \max\{\deg_{\langle R\rangle} v, \deg_{\langle R\rangle} w\}$. Then $\ell(Q) \geqslant d$. Thus $Q$ contains at least $d + 1$ vertices. Going around $P$, let there be $t$ occurrences of a vertex $y$ joined to one of $v$ or $w$ and followed (not necessarily immediately) by a vertex $z$ joined to the other of $v$ and $w$; then there are at least $d + 1$ vertices between $y$ and $z$ on $P$ which are joined to neither $v$ nor $w$, for otherwise $P$ could be extended. Thus $2n - r - a = \ell(P) \geqslant$ number of edges from $v$ to $P$ + number of edges from $w$ to $P$ + number of vertices of $P$ joined to $v$ and/or $w$ $+ t(d - 1) \geqslant 3n - 3k - 3d + td - t$. Since $v$ and $w$ are both joined to vertices of $P$, $t \geqslant 2$. Further, $1 \leqslant d \leqslant r - 1$. Thus $1 - d \leqslant 0$. It follows that $n \leqslant 3k + 1 - a$. But $n \geqslant 3k + 2 - a$, so $\ell(Q) = 0$ and $R$ is independent.

Now we fix $v$ and let $A = A_v$, $B = B_v$, and $C = C_v$, Let $X = V(P) - (A \cup B \cup C)$ and let $s = |A - B| = |B - A|$. It is easy to see that $s \geqslant 1$ when $k \geqslant 3$. By Lemma 2, $|A| = |B| = |C| = n - k$ and $|X| = 2k - (r + s) - a$. Since $|X| \geqslant 0$, $r + s \leqslant 2k - a$.

LEMMA 3. *If* $n \geqslant 3k + 2 - a$, *then* $r \leqslant k - a$.

*Proof.* Let $d$ be the number of edges from $R$ to $B$. Then $d \leqslant r - 1$ by Lemma 1. Also by Lemma 1, $B$ is independent. Thus there are $(n - k)(n - k + r) - 2d$ edges from $R \cup B$ to the other $n + k - r - a$ vertices of $G$. Since $G$ has $(2n - a)(n - k)/2$ edges, $(n - k)(n - k + r) - 2d + d \leqslant (2n - a)(n - k)/2$, from which we get $r \leqslant k - \frac{1}{2}a + (k - \frac{1}{2}a - 1)/(n - k - 1)$. Since $r$ is an integer and $n \geqslant 3k + 2 - a$, $r \leqslant k - a$.

LEMMA 4. *If* $n \geqslant k^2 + k + 1$, *then* $r + s \leqslant k$.

*Proof.* Suppose $r + s > k$. By Lemmas 1 and 2, $|E(\langle A \cup B \cup R\rangle)| \leqslant s^2 + 2(r - 1)$. Since $|A \cup B \cup R| = n - k + r + s$, there are at least $(n - k + r + s)(n - k) - 2(s^2 + 2r - 2)$ edges from $A \cup B \cup R$ to $C \cup X$; further, $|C \cup X| = n + k - r - s - a$. Thus

$$(n - k + r + s)(n - k) - 2(s^2 + 2r - 2) \leqslant (n + k - r - s - a)(n - k),$$

whence (using the assumption that $r + s \geqslant k + 1$),

$$n \leqslant k + [(s^2 + 2r - 2)/(r + s - k + \tfrac{1}{2}a)].$$

Denoting this upper bound for $n$ by $f(a, k, r, s)$, holding $a$, $k$, and $r$ constant, and recalling that $k + 1 - r \leqslant s \leqslant 2k - a - r$, we find that $f(a, k, r, k + 1 - r)$ is a maximum for $f$ except when $a = 1$ and the pair $(k, r)$ is in $\{(3, 1), (3, 2), (4, 2), (4, 3), (5, 3), (5, 4)\}$. But in these exceptional cases, $f(1, k, r, s) \leqslant k^2 + k$. Further, in all other cases as $r$ ranges through $[1, k]$, treating the cases $a = 0$ and $a = 1$ separately and holding $k$ constant, we get $f(a, k, r, s) \leqslant k^2 + k$. The lemma follows.

LEMMA 5. *Let $X_0$ be the subset of $X$ such that the elements of $X_0$ are adjacent to no vertices of $A \cap B$. Then*

(1) *if* $a = 0$, $|X_0| \geqslant k - r - s + 1$; *and*
(2) *if* $a = 1$, $|X_0| \geqslant k - r - s$.

*Proof.* There are $s$ intervals on $P$ in which vertices of $X$ might be found. Number these intervals as $1, 2, \dots, s$ with $m_i$ elements of $X$ in interval $i$ in such a way that $m_1, m_2, m_3, \dots, m_e$ are even and $m_{e+1}, m_{e+2}, \dots, m_s$ are odd, with $e \geqslant 0$. It is easily seen that if two vertices of $X$ which are successive around $P$ are both joined to elements of $A \cap B$, then there is a cycle of $G$ larger than $P$. Hence at least the smallest number of nonconsecutive elements of the sequence of vertices in $X$ in interval $i$, or $\{(m_i - 1)/2\}$, are not joined to any vertex in $A \cap B$. Thus

$$|X_0| \geqslant \sum_{i=1}^{e}\left(\frac{m_i - 1}{2} + \frac{1}{2}\right) + \sum_{i=e+1}^{s}\frac{m_i - 1}{2}$$
$$= \frac{1}{2}|X| - \frac{1}{2}(s - e) \geqslant \frac{1}{2}(2k - r - 2s - a).$$

If $a = 0$, $|X_0| \geqslant k - r - s + \frac{1}{2}r \geqslant k - r - s + \frac{1}{2}$ since $r \geqslant 1$. But $|X_0|$, $k$, $r$, and $s$ are integers, so $|X_0| \geqslant k - r - s + 1$. If $a = 1$, $|X_0| \geqslant k - r - s + \frac{1}{2}r - 1 \geqslant k - r - s - \frac{1}{2}$, whence $|X_0| \geqslant k - r - s$.

THEOREM. *Suppose $k \geqslant 3$. Then $G$ is Hamiltonian if*

(a) $a = 0$ *and* $n \geqslant k^2 + k + 1$, *or*
(b) $a = 1$ *and* $n \geqslant 2k^2 - 3k + 3$.

*Proof.* Suppose $G$ is not Hamiltonian. By Lemma 4, $r + s \leqslant k$. By Lemma 5 and the definitions, $|A \cup B \cup R \cup X_0| \geqslant n + 1 - a$. Choose a subset $X_0'$ of $X_0$ such that $|A \cup B \cup R \cup X_0'| = n + 1 - a$. By the definitions and Lemmas 1 and 2, we have at most

| | edges from | to |
|---|---|---|
| $s^2$ | $A$ | $B$ |
| $r - 1$ | $A$ | $R$ |
| $s(k - r - s + 1 - a)$ | $A$ | $X_0'$ |
| $r - 1$ | $B$ | $R$ |
| $s(k - r - s + 1 - a)$ | $B$ | $X_0'$ |
| $(r - 1)(k - r - s + 1 - a)$ | $R$ | $X_0'$ |
| $\frac{1}{2}(k - r - s - a)(k - r - s + 1 - a)$ | $X_0'$ | $X_0'$ |

in $G$ and no other edges in $\langle A \cup B \cup R \cup X_0' \rangle$. Thus there are at least

$$(n+1-a)(n-k) - 2\,\{s^2 + 2r - 2 + 2s(k-r-s+1-a) + (r-1)(k-r-s+1-a) + \tfrac{1}{2}(k-r-s-a)(k-r-s+1-a)\}$$

edges from $A \cup B \cup R \cup X_0'$ to $(C \cup X) - X_0'$. Since this number is less than or equal to $(n-1)(n-k)$, we get

$$n \leqslant k + \frac{2}{2-a}\Big\{(r+s-r-1)^2 + 2(r+s) - 3 + (k-(r+s)+1-a)\Big(2(r+s) - r - 1 + \frac{k-(r+s)-a}{2}\Big)\Big\}.$$

Since $r \geqslant 1$, and replacing $r + s$ by $t$ which now ranges in $[2, k]$,

$$n \leqslant k + \frac{2}{2-a}\Big\{(t-2)^2 + 2t - 3 + (k-t+1-a)\Big(2t - 2 + \frac{k-t-a}{2}\Big)\Big\}.$$

Routine manipulation now shows that if $a = 0$, then $n \leqslant k^2 + k - 1$, while if $a = 1$, then $n \leqslant 2k^2 - 3k + 2$. Since $n$ exceeds the specified bound in each case, $G$ is Hamiltonian.

Non-Hamiltonian graphs satisfying the conditions of regularity of degree $n - k$ with $2n$ or $2n - 1$ vertices, and two connectedness, are known. For example, choose graphs $H_1'$, $H_2'$, and $H_3'$ such that $H_i'$ is isomorphic to $K_{2t}$. In $V(H_i')$, choose disjoint sets $A_i$ and $B_i$, each of cardinality $2t/3 - [i/3]$, and form $H_i$ from $H_i'$ by deleting from $H_i'$ a matching, each of whose edges joins a member of $A_i$ to a member of $B_i$. Form a graph $H$ by joining a new vertex $u$ to every member of every $A_i$ and a new vertex $v$ to every member of every $B_i$. Then, letting $k = t + 2$ and $n = 3k - 5$, $H$ is non-Hamiltonian, has $2n$ vertices, and is two connected and regular of degree $n - k$. Many other similar examples can be constructed. Thus the theorem clearly requires some lower bound for $n$. But this lower bound surely is not as large as the ones used here.

## References


1. M. Behzad and G. Chartrand, "Introduction to the Theory of Graphs," Allyn and Bacon, Boston, 1971.
2. G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* **2** (1952), 69–81.
3. P. Erdös and A. M. Hobbs, A class of Hamiltonian regular graphs, *J. Graph Theory*, to appear.
4. L. Gordon, Hamiltonian circuits in graphs with many edges, unpublished.